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# Wavelets and Quasiasymptotics at a Point

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Let $\{V_j, j \in \mathbf{Z}\}$ be a MRA of the space $L^2(\mathbf{R})$, $h$ a tempered distribution, and $h_j$ its projection to $V_j$, $j \in \mathbf{Z}$. Then we prove that if $h$ has the quasiasymptotic behavior at zero related to a regularly varying function, then so does each $h_j$, $j \in \mathbf{Z}$, and also prove, with an additional condition, the opposite statement. © 1999 Academic Press

## 1. INTRODUCTION

The problem of convergence of multiresolution and wavelet expansions was studied by several authors, let us mention [ME], [KKR], [W1]. In [ME], [KKR] such expansions were observed in some well known spaces of functions and distributions ($L^p$ spaces, Sobolev spaces, Hölder spaces, etc.), and in [W1, W2] G. G. Walter studied spaces of tempered distributions of order $r-1$, $\mathcal{S}'_{r-1}$, the dual space of the space

$$\mathcal{S}_{r-1} = \left\{ f \in C^{r-1} \mid \left| \left(\frac{d}{dx}\right)^q f(x) \right| \leqslant C_m (1+|x|)^{-m}, \right.$$
$$\left. q = 0, ..., r-1, m = 0, 1, ..., x \in \mathbf{R} \right\}.$$

Let us recall his main theorem concerning the value of a distribution at a point.

Following Lojasiewicz [L], we say that a tempered distribution $h$ has a value $\gamma$ of order $r$ at a point $x_0$ if there exists a continuous function $H(x)$ of polynomial growth such that $D^r H = h$ in some neighborhood of $x_0$ and

$$\lim_{x \to x_0} \frac{H(x)}{(x-x_0)^r} = \frac{\gamma}{r!}.$$

Let then $h \in \mathcal{S}'_{r-1}$ and let $V_j$, $j \in \mathbf{N}$, be a nested sequence that forms an $r$-regular multiresolution approximation of $L^2(\mathbf{R})$. By $h_j$ we denote the



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orthogonal projection of $h$ onto $V_j$. (The notion of multiresolution approximation and some details on orthogonal projections will be given in the next section.) Using quasi-positive delta sequences [W1, pp. 110–112], Walter proved the following

THEOREM 1 [W1, W2]. *Let $h \in \mathcal{S}'_{r-1}$ have a value $\gamma$ of order $\alpha \leqslant r$ at $x = x_0$. Then the orthogonal projections $h_j$ of $h$ onto the spaces $V_j$ satisfy*

$$h_j(x_0) \to \gamma \qquad \text{as} \quad j \to \infty.$$

A more general notion than the distributional value at a point was introduced by the Russian mathematician B. I. Zavialov in 1973, namely the quasiasymptotic behavior at some point or infinity [VDZ], [PST]. In the rest of the section we define the quasiasymptotics at zero and give some motivation for our investigation. We say that $h \in \mathcal{S}'$ has quasiasymptotics at zero (in $\mathcal{S}'$) related to a continuous positive function $c(\varepsilon)$, if there exists $g \in \mathcal{S}'$, $g \neq 0$, such that

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{h(\varepsilon x)}{c(\varepsilon)}, \sigma(x) \right\rangle = \langle g(x), \sigma(x) \rangle, \qquad \sigma \in \mathcal{S}.$$

Note that if we put $c(\varepsilon) = 1$, we obtain the Lojasiewicz value at a zero. The quasiasymptotic behaviour turned out to be more appropriate for the Abelian and Tauberian type theorems for several integral transforms, such as Fourier, Laplace, Stieltjes, and Mellin transform, than some other types of asymptotic behaviors of a distribution, including the Lojasiewicz value at a point. We can therefore use quasiaymptotics to obtain the Abelian and Tauberian theorems for wavelet transform. B. I. Zavialov applied the quasiasymptotic behavior of distributions in the study of asymptotic properties of form-factors and the Jost–Lemann–Dyson spectral function [Z]. For more details on various definitions of asymptotic behaviors and their applications in PDEs and mathematical physics (especially in the quantum field theory) see [VDZ], [PST] and the references cited therein.

It is known that some singular functions and distributions have quasiasymptotic behavior different from their classical asymptotic behavior, or the last does not exist at all (e.g., $e^{ix}$ at infinity, delta distribution as well as its derivatives). The quasiasymptotic behavior is essentially characterized by the behavior of an appropriate integral transform, which can even be analytic function, although the original function or distribution is singular (more examples can be found in [VDZ], [PST]). Since the already mentioned projections, $h_j$, are a kind of integral transform, it is of interest to relate them to the quasiasymptotic behavior.

The goal of this paper is to prove that in the above theorem one can replace the Lojasiewicz value at a point with the quasiasymptotic behavior related to a regularly varying function (see [S]). Namely, we shall prove that if $h \in \mathscr{S}'$ has a given quasiasymptotic behavior at (say) zero, then its projections to $V_j$, $h_j$, have the same quasiasymptotics related to the same regularly varying function. This is the extension of Walter's theorem cited above. Moreover, with an additional condition, we state that the converse also holds.

## 2. MULTIRESOLUTION APPROXIMATION AND EXPANSION

The notion of multiresolution approximation was introduced in [MA] as a natural approach to the wavelet orthonormal bases. One can easily obtain a wavelet basis associated to the particular multiresolution approximation as follows.

DEFINITION 1 [ME]. A *multiresolution approximation of* $L^2(\mathbf{R})$ (shortly MRA) is, by definition, an increasing sequence of closed linear subspaces $V_j$, $j \in \mathbf{Z}$, of $L^2(\mathbf{R})$, with the properties

$$\bigcap_{-\infty}^{\infty} V_j = \{0\} \qquad \text{and} \qquad \bigcup_{-\infty}^{\infty} V_j \text{ is dense in } L^2(\mathbf{R}); \tag{1}$$

$$\text{for all } f \in L^2(\mathbf{R}) \text{ and all } j \in \mathbf{Z}, \qquad f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}; \tag{2}$$

$$\text{for all } f \in L^2(\mathbf{R}) \text{ and all } k \in \mathbf{Z}, \qquad f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0; \tag{3}$$

$$\begin{aligned}&\text{there exists a function } \phi \in V_0, \text{ such that the sequence}\\ &\{\phi(x-k), k \in \mathbf{Z}\} \text{ is an orthonormal basis of the space } V_0.\end{aligned} \tag{4}$$

The function $\phi$ given by (4) is called *the scaling function* (some authors call it *father wavelet* also). We say that a multiresolution approximation, $V_j$, $j \in \mathbf{Z}$, is *r-regular* ($r \in \mathbf{N}$), if the scaling function $\phi$ fulfills the following additional condition.

For all $0 \leqslant q \leqslant r$ and all $m \in \mathbf{N}$, there exists a constant $C_m$ such that

$$\left| \left( \frac{d}{dx} \right)^q \phi(x) \right| \leqslant C_m (1 + |x|)^{-m}, \qquad x \in \mathbf{R}. \tag{5}$$

It is well known that for every $r \in \mathbf{N}$, there exists an $r$-regular MRA, i.e., a function $\phi$ which satisfies the conditions (4) and (5). This fact will enable us to analyze the space $\mathscr{S}'$, since $\mathscr{S}' = \bigcup_{r \in \mathbf{N}} \mathscr{S}'_r$. In fact, we shall use a family of infinitely smooth scaling functions.

Let $V_j$, $j \in \mathbf{Z}$, be an $r$-regular MRA of $L^2(\mathbf{R})$. The orthogonal complement of $V_j$ in $V_{j+1}$ is denoted by $W_j$. The space $L^2(\mathbf{R})$ can be then observed as a direct sum of subspaces $W_j$, $j \in \mathbf{Z}$. Using the scaling function $\phi$ (see (4)) one can construct a function $\psi \in W_0$ with the properties

for all $0 \leqslant q \leqslant r$, $m \geqslant 1$ there exists a constant $C_m$ such that

$$\left|\left(\frac{d}{dx}\right)^q \psi(x)\right| \leqslant C_m(1+|x|)^{-m}, \qquad x \in \mathbf{R} \quad \text{holds, and} \tag{6}$$

the sequence $\psi(x-k)$, $k \in \mathbf{Z}$, is an orthonormal basis of $W_0$ (7)

(see [ME, pp. 72–81] for details). The collection $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \mid j \in \mathbf{Z}, k \in \mathbf{Z}\}$ is an orthonormal basis of the space $L^2(\mathbf{R})$. The functions $\psi_{j,k}$, $j, k \in \mathbf{Z}$, are called *wavelets* of class $r$, associated to the given $r$-regular MRA. It can be shown that for the Fourier transforms $\hat{\phi}$ and $\hat{\psi}$ of the scaling function $\phi$ and the corresponding wavelet $\psi$, respectively, the following relation holds

$$\hat{\psi}(\xi) = ((\hat{\phi}(\xi/2))^2 - (\hat{\phi}(\xi))^2)^{1/2}\, e^{-i\xi/2}. \tag{8}$$

Y. Meyer proved that there exist orthonormal wavelet bases with infinitely smooth basic wavelets (see [ME]). In that case, the support of the function $\psi$ must be the whole real line. In most applications it is usually convenient to work with compactly supported wavelets. In [D] I. Daubechies showed that for an arbitrarily nonnegative integer $r$, there exists an $r$-regular MRA of $L^2(\mathbf{R})$ such that the corresponding functions $\phi$ and $\psi$ have compact supports. In the proof of this important theorem she gave the construction of such wavelets.

The wavelet bases are not only orthonormal bases of $L^2(\mathbf{R})$, but the unconditional bases for the spaces $L^p$, $1 < p < \infty$, Sobolev spaces, and Hölder spaces as well.

However, in this paper, the properties of scaling functions will play the essential role.

Let there be given an $r$-regular MRA of $L^2(\mathbf{R})$ and let $\phi$ be a scaling function with properties (4) and (5). The operator $E_0$ of orthogonal projection from $L^2(\mathbf{R})$ onto the subspace $V_0$ is defined by the kernel

$$E(x, y) = \sum_{k \in \mathbf{Z}} \phi(x-k)\, \phi(y-k)$$

in the following way

$$E_0 h(x) = \langle h(y), E_0(x, y)\rangle = \int E(x, y)\, h(y)\, dy.$$

The kernel of the projection operator onto the subspace $V_j$ will then be

$$E_j(x, y) = 2^j E(2^j x, 2^j y). \tag{9}$$

Thus the projection of the function $h \in L^2(\mathbf{R})$ onto the subspace $V_j$ is given by

$$E_j h(x) = \langle h(y), E_j(x, y) \rangle = \int h(y)\, E_j(x, y)\, dy. \tag{10}$$

The functions $E_j$ are the reproducing kernels for $V_j$, $j \in \mathbf{Z}$, i.e., for $h \in V_j$ it holds $E_j h(x) = h(x)$. We will use the notation $h_j(x) = E_j h(x)$. In the last section, we shall be only interested in properties of the kernel of the integral transform (10). Thus we shall leave the MRA frame and allow $j$ to be a real number, not necessarily an integer.

From the definition of the kernel $E(x, y)$ and the properties of the scaling function $\phi$ it follows

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} E(x, y) \right| \leqslant C_m (1 + |x - y|)^{-m}, \qquad \text{for all} \quad m \in \mathbf{N}, \tag{11}$$

where $\alpha$ and $\beta$ in (11) are nonnegative integers less than or equal to $r$ (in an $r$-regular MRA); $E(x + k, y + k) = E(x, y)$ for all $k \in \mathbf{Z}$, and we have the symmetry $E(x, y) = E(y, x)$. As indicated in [ME, pp. 33–38; W2, pp. 40–43], it holds

$$\int E(x, y)\, x^\alpha\, dx = y^\alpha, \tag{12}$$

for all nonnegative integers $0 \leqslant \alpha \leqslant r$, thus it is possible to observe more general MRAs than the MRA of $L^2(\mathbf{R})$. Namely, from (12) it follows that if the scaling function $\phi$ belongs to $\mathcal{S}_r$, all polynomials up to the order $r$ belong to $V_0$, and therefore to $V_j$, $j \geqslant 0$.

G. G. Walter proved that the sequence of the reproducing kernels $\{E_j\}$, in an $r$-regular MRA, is a quasi-positive delta sequence. Then it follows

PROPOSITION 1 [W2]. *Let $h$ be a function in $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$ continuous on $(a, b)$ and let $h_j$ be the projections of the function $h$ onto the $V_j$. Then $h_j \to h$ when $j \to \infty$ uniformly on compact subsets of $(a, b)$.*

Following [KKR, p. 89], the sequence of projections $(h_j)_{j \in \mathbf{Z}}$ will be called *multiresolution expansion* of $h$, where $h$ can be an element of the space of tempered distributions.

## 3. QUASIASYMPTOTICS AT ZERO

All the definitions and properties in this section can be found in [VDZ], [PST].

DEFINITION 2. *Let $h \in \mathscr{S}'$ and let $c(x), x \in (0, a), a > 0$, be a continuous positive function. We say that $h$ has the quasiasymptotics at zero (in $\mathscr{S}'$) related to $c(\varepsilon)$, if there exists $g \in \mathscr{S}'$, $g \neq 0$, such that*

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{h(\varepsilon x)}{c(\varepsilon)}, \sigma(x) \right\rangle = \langle g(x), \sigma(x) \rangle, \qquad \sigma \in \mathscr{S}.$$

*In this case we write $h \overset{q}{\sim} g$ at 0 related to $c(\varepsilon)$ in $\mathscr{S}'$.*

This definition can be extended to the space of distributions $\mathscr{D}'$. The relation between the quasiasymptotics at zero in $\mathscr{S}'$ and in $\mathscr{D}'$ is studied in [P].

PROPOSITION 2. *Let $h$ and $c$ satisfy the conditions of Definition 2. Then for some $\nu \in \mathbf{R}$ and some slowly varying function $L$ at 0, $c(x) = x^\nu L(x)$, $x \in (0, a)$. Moreover, $g$ is homogenous with order of homogeneity $\nu$, i.e., $g(mx) = m^\nu g(x)$, $m > 0$, $x \in \mathbf{R}$.*

Recall that the function $L$: $(0, a) \mapsto \mathbf{R}^+$, $a > 0$, is *slowly varying* at 0 if for all $\lambda > 0$

$$\lim_{\varepsilon \to 0^+} \frac{L(\lambda\varepsilon)}{L(\varepsilon)} = 1.$$

A measurable function $\rho$: $(0, a) \mapsto \mathbf{R}^+$, $a > 0$, is *regularly varying at* 0 if there exists $\alpha \in \mathbf{R}$, such that for all $\lambda > 0$

$$\lim_{\varepsilon \to 0^+} \frac{\rho(\lambda\varepsilon)}{\rho(\varepsilon)} = \lambda^\alpha.$$

A function is regularly varying if it can be written as $\rho(x) = x^\alpha L(x)$, $x > a$, for some $\alpha \in \mathbf{R}$ and some slowly varying function $L$ at 0. The previous proposition claims that $c(x)$ is a regularly varying function.

If $\nu = 0$ and $L \equiv 1$ the definition of quasiasymptotics at zero in $\mathscr{D}'$ is a slight generalization of the above mentioned Lojasiewicz definition of the distributional "value at 0" [L], which can be also seen from the following characterization.

THEOREM 2 [PST]. *Let $h \in \mathscr{D}'$ have the quasiasymptotics at 0 related to $\varepsilon^\nu L(\varepsilon)$. If $\nu > 0$ (or, if $\nu > -1$ and $L$ is bounded on some interval $(0, a)$ $a > 0$),* *then there are a continuous function H defined on* $(-1, 1)$, *an integer m, and* $(C_+, C_-) \neq (0, 0)$ *such that* $H^{(m)}(x) = h(x)$ *and*

$$\lim_{x \to \pm 0} \frac{H(x)}{|x|^{\nu+m} L(|x|)} = C_{\pm}.$$

The quasiasymptotics at an arbitrary finite point can be defined and treated in a similar way. Thus it is enough to study only the point $x=0$ (see the first remark at the end of the paper).

## 4. MAIN RESULTS

From now on, we assume that $\rho$ is a regularly varying function at zero. The main results of this paper are given in the following two theorems.

THEOREM 3. *Let a distribution* $h \in \mathscr{S}'$ *have the quasiasymptotics at zero (in* $\mathscr{S}'$*) related to* $\rho(x)$ *equal to* $\gamma(x) \neq 0$ ($h \overset{q}{\sim} \gamma$). *Then* $h_j(x) = \langle h(y), E_j(x, y) \rangle$, $j \in \mathbf{R}$, *have the quasiasymptotics at zero (in* $\mathscr{S}'$*) related to* $\rho(x)$ *equal to* $\gamma(x) \neq 0$ ($h_j \overset{q}{\sim} \gamma$) *also.*

THEOREM 4. *Let the functions* $h_j(x) = \langle h(y), E_j(x, y) \rangle$, $j \in \mathbf{R}$, *have the quasiasymptotics at zero equal to* $\gamma_j$, *and let* $\gamma_j \to \gamma \neq 0$ *as j tends to infinity. Moreover assume that the family* $\{h(\varepsilon y)/\rho(\varepsilon) \mid \varepsilon \in (0, 1)\}$ *is bounded. Then h has the quasiasymptotics at zero equal to* $\gamma$.

In order to prove these theorems, we need the following lemma. To that end, put $E_{\varphi(j,\varepsilon)}(x, y) = 2^j \varepsilon\, E(2^j \varepsilon x, 2^j \varepsilon y)$, where $\varphi(j, \varepsilon) = j + \log_2 \varepsilon$.

LEMMA 1. *The family*

$$\{\langle E_{\varphi(j,\varepsilon)}(x, y), \sigma(x) \rangle, \varepsilon \in (0, 1); \sigma \in \mathscr{S}\}$$

*is, for every* $j \in \mathbf{Z}$, *bounded in* $\mathscr{S}$, *uniformly in* $\varepsilon$.

*Proof of the Lemma.* For a given $\sigma \in \mathscr{S}$ there exists $r$ such that $\sigma \in \mathscr{S}_r$, and for that $r$ we choose an $r$-regular MRA. The lemma will be proved if we show that the expression

$$\sup_{y \in \mathbf{R}} \left| (1 + y^2)^{m/2} \left( \frac{d^p}{dy^p} \int_{-\infty}^{\infty} E_{\varphi(j,\varepsilon)}(x, y)\, \sigma(x)\, dx \right.\right.$$
$$\left.\left. - \sigma(y) \frac{d^p}{dy^p} \int_{-\infty}^{\infty} E_{\varphi(j,\varepsilon)}(x, y)\, dx \right) \right|$$

is finite for all $k$ and every $m, p \leqslant k$. For that reason we consider the expression

$$\begin{aligned}(1+y^2)^{m/2}\,\bigg|\frac{d^p}{dy^p}\int_{-\infty}^{y-c} E_{\varphi(j,\varepsilon)}(x,y)\,\sigma(x)\,dx\\
+\frac{d^p}{dy^p}\int_{y-c}^{y+c} E_{\varphi(j,\varepsilon)}(x,y)(\sigma(x)-\sigma(y))\,dx\\
+\frac{d^p}{dy^p}\int_{y+c}^{\infty} E_{\varphi(j,\varepsilon)}(x,y)\,\sigma(x)\,dx\\
-\sigma(y)\frac{d^p}{dy^p}\left(\int_{-\infty}^{y-c} E_{\varphi(j,\varepsilon)}(x,y)\,dx+\int_{y+c}^{\infty} E_{\varphi(j,\varepsilon)}(x,y)\,dx\right)\bigg|\\
\leqslant (1+y^2)^{m/2}\,(|I_1|+|I_2|+|I_3|+|I_4|+|I_5|).\end{aligned}$$

First, we show that the $(1+y^2)^{m/2}\,|I_3|$ is bounded:

$$\begin{aligned}(1+y^2)^{m/2}\left|\frac{d^p}{dy^p}\int_{y+c}^{\infty} E_{\varphi(j,\varepsilon)}\,(x,y)\,\sigma(x)\,dx\right|\\
\leqslant C_s(1+y^2)^{m/2}\int_{y+c}^{\infty}\frac{|(d^p/dy^p)\,E_{\varphi(j,\varepsilon)}(x,y)|}{(1+|x|)^s}\,dx\\
\leqslant C_s\frac{(1+y^2)^{m/2}}{(1+|y+c|)^s}\int_{y+c}^{\infty}\left|\frac{d^p}{dy^p}E_{\varphi(j,\varepsilon)}(x,y)\right|dx.\end{aligned}$$

Since $s$ can be chosen arbitrarily, it remains to show that

$$\int_{y+c}^{\infty}\left|\frac{d^p}{dy^p}E_{\varphi(j,\varepsilon)}(x,y)\right|dx$$

is bounded. After the change of variables $t=2^j\varepsilon x$, we get

$$\begin{aligned}\int_{2^j\varepsilon(y+c)}^{\infty}\left|\frac{d^p}{dy^p}E(t,2^j\varepsilon y)\right|dt \leqslant \int_{2^j\varepsilon(y+c)}^{\infty}\sum_{k\in\mathbf{Z}}|\phi(t-k)|\left|\frac{d^p}{dy^p}\,\phi(2^j\varepsilon y-k)\right|dt\\
\leqslant \sum_{k\in\mathbf{Z}}\left|\frac{d^p}{dy^p}\phi(2^j\varepsilon y-k)\right|\int_{2^j\varepsilon(y+c)}^{\infty}|\phi(t-k)|\,dt. \qquad (13)\end{aligned}$$

Using the estimate

$$\int_{2^j\varepsilon(y+c)}^{\infty} |\phi(t-k)|\,dt \leqslant \int_{2^j\varepsilon(y+c)}^{\infty} \frac{C_s}{(1+|t-k|)^{s+2}}\,dt$$
$$\leqslant \frac{C_s}{(1+|2^j\varepsilon(y+c)-k|)^s} \int_{2^j\varepsilon(y+c)}^{\infty} \frac{dt}{(1+|t|)^2}$$
$$\leqslant \frac{C}{(1+|2^j\varepsilon(y+c)-k|)^s}$$

we see that (13) is less than or equal to

$$\sum_{k\in\mathbf{Z}} \left|\frac{d^p}{dy^p}\,\phi(2^j\varepsilon y-k)\right| \frac{C}{(1+|\,2^j\varepsilon(y+c)-k|)^s}$$

$$\leqslant |2^j\varepsilon|^p\, C_1 \sum_{k\in\mathbf{Z}} \frac{1}{(1+|2^j\varepsilon y-k|)^s}\, \frac{1}{(1+|2^j\varepsilon(y+c)-k|)^s}.$$

Since the last series is uniformly convergent (see [W2, p. 122]) we see that $(1+y^2)^{m/2}\,|I_3|$ is, for fixed $j$, uniformly bounded in $\varepsilon$. The boundedness of the sum

$$(1+y^2)^{m/2}\,(|I_1|+|I_4|+|I_5|)$$

can be obtained in a similar way. It remains to show that the product $(1+y^2)^{m/2}\,|I_2|$ is bounded too.

$$(1+y^2)^{m/2}\left|\frac{d^p}{dy^p}\int_{y-c}^{y+c} E_{\varphi(j,\varepsilon)}(x,y)(\sigma(x)-\sigma(y))\,dx\right|$$
$$\leqslant (1+y^2)^{m/2}\left|\frac{d^p}{dy^p}\int_{y-c}^{y+c} E_{\varphi(j,\varepsilon)}(x,y)\,\sigma'(\xi)(x-y)\,dx\right|$$
$$\leqslant (1+y^2)^{m/2}\,c\,|\sigma'(\xi)| \int_{y-c}^{y+c}\left|\frac{d^p}{dy^p}E_{\varphi(j,\varepsilon)}(x,y)\right|dx$$
$$\leqslant cC_s\,\frac{(1+y^2)^{m/2}}{(1+|y-c|)^s}\int_{y-c}^{\infty}\left|\frac{d^p}{dy^p}E_{\varphi(j,\varepsilon)}(x,y)\right|dx<\infty,$$

where $\xi\in(y-c,y+c)$, $s>m$, and we have already shown that the last integral is bounded. The proof is completed. ▮

*Proof of Theorem* 3. Let $\lim_{\varepsilon\to 0^+}\langle h(\varepsilon x)/\rho(\varepsilon),\sigma(x)\rangle = \langle \gamma(x),\sigma(x)\rangle$, for all $\sigma\in\mathscr{S}$. Then we have

$$\left\langle \frac{h_j(\varepsilon x)}{\rho(\varepsilon)}, \sigma(x) \right\rangle = \left\langle \frac{\langle h(y), E_j(\varepsilon x, y)\rangle}{\rho(\varepsilon)}, \sigma(x) \right\rangle = \left\langle \frac{h(y)}{\rho(\varepsilon)}, \langle E_j(\varepsilon x, y), \sigma(x)\rangle \right\rangle$$

$$= \left\langle \frac{h(\varepsilon y)}{\rho(\varepsilon)}, \langle E_{\varphi(j,\varepsilon)}(x, y), \sigma(x)\rangle \right\rangle.$$

Using the previous lemma and the equivalence of weak and strong convergence in $\mathscr{S}'$ we conclude that

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{h(\varepsilon y)}{\rho(\varepsilon)}, \langle E_{\varphi(j,\varepsilon)}(x, y), \sigma(x)\rangle \right\rangle = \langle \gamma(x), \sigma(x)\rangle$$

that is,

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{h_j(\varepsilon x)}{\rho(\varepsilon)}, \sigma(x) \right\rangle = \langle \gamma(x), \sigma(x)\rangle, \qquad \sigma \in \mathscr{S}.$$

The theorem is proved. ∎

*Proof of Theorem* 4. Let us put now $\eta(\varepsilon) = 1/\varepsilon + \log_2 \varepsilon$, that is,

$$E_{\eta(\varepsilon)}(x, y) = 2^{1/\varepsilon}\, \varepsilon E(2^{1/\varepsilon}\varepsilon x, 2^{1/\varepsilon}\varepsilon y),$$

and let $\lim_{\varepsilon \to 0^+} \langle h_j(\varepsilon x)/\rho(\varepsilon), \sigma(x)\rangle = \langle \gamma_j(x), \sigma(x)\rangle$, $\sigma \in \mathscr{S}$. For

$$h_{1/\varepsilon}(x) = \int_{\mathbf{R}} h(y)\, E_{1/\varepsilon}(x, y)\, dy,$$

using the boundedness of the family $\{h(\varepsilon y)/\rho(\varepsilon) \mid \varepsilon \in (0, 1)\}$, we have

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{h_{1/\varepsilon}(\varepsilon x)}{\rho(\varepsilon)}, \sigma(x) \right\rangle = \langle \gamma(x), \sigma(x)\rangle, \qquad \sigma \in \mathscr{S}.$$

Since

$$\left\langle \frac{\langle h(y), E_{1/\varepsilon}(\varepsilon x, y)\rangle}{\rho(\varepsilon)}, \sigma(x) \right\rangle$$

$$= \left\langle \frac{h(\varepsilon y)}{\rho(\varepsilon)}, \sigma(y) \right\rangle + \left\langle \frac{h(\varepsilon y)}{\rho(\varepsilon)}, \langle E_{\eta(\varepsilon)}(x, y) - \delta(x - y), \sigma(x)\rangle \right\rangle$$

the theorem will be proved once we show that the expression

$$(1 + |y|^2)^{m/2} \left| \frac{d^p}{dy^p} \langle E_{\eta(\varepsilon)}(x, y) - \delta(x - y), \sigma(x)\rangle \right|$$

tends to zero when $\varepsilon \to 0$. By using the equality $(d^p/dy^p)\, E_{\eta(\varepsilon)}\,(x, y) = (d^p/dx^p)\, E_{\eta(\varepsilon)}(y, x)$ we get

$$
\begin{aligned}
&(1+|y|^2)^{m/2} \left| \int_{\mathbf{R}} \frac{d^p}{dy^p} E_{\eta(\varepsilon)}(x, y)\, \sigma(x)\, dx - (-1)^p\, \sigma^{(p)}(y) \int_{\mathbf{R}} E_{\eta(\varepsilon)}(x, y)\, dx \right| \\
&\quad = (1+|y|^2)^{m/2} \left| (-1)^p \left( \int_{\mathbf{R}} E_{\eta(\varepsilon)}(y, x) \frac{d^p}{dx^p} \sigma(x)\, dx \right.\right. \\
&\qquad \left.\left. - \sigma^{(p)}(y) \int_{\mathbf{R}} E_{\eta(\varepsilon)}(y, x)\, dx \right) \right| \\
&\quad \leqslant (1+y^2)^{m/2} \left| \int_{-\infty}^{y-c} E_{\eta(\varepsilon)}(y, x) \frac{d^p}{dx^p} \sigma(x)\, dx \right. \\
&\qquad + \int_{y-c}^{y+c} E_{\eta(\varepsilon)}(y, x)(\sigma^{(p)}(x) - \sigma^{(p)}(y))\, dx + \int_{y+c}^{\infty} E_{\eta(\varepsilon)}(y, x) \frac{d^p}{dx^p} \sigma(x)\, dx \\
&\qquad \left. - \sigma^{(p)}(y) \left( \int_{-\infty}^{y-c} E_{\eta(\varepsilon)}\,(x, y)\, dx + \int_{y+c}^{\infty} E_{\eta(\varepsilon)}\,(x, y)\, dx \right) \right| \\
&\quad = (1+y^2)^{m/2}\, (|I_1| + |I_2| + |I_3| + |I_4| + |I_5|).
\end{aligned}
$$

We shall first estimate $(1+y^2)^{m/2}\, |I_3|$.

$$
\begin{aligned}
&(1+y^2)^{m/2}\, |I_3| \\
&\qquad \leqslant C_s \frac{(1+y^2)^{m/2}}{(1+|y+c|)^s} \int_{y+c}^{\infty} |E_{\eta(\varepsilon)}(y, x)|\, dx \\
&\qquad \leqslant C_1 \sum_{k \in \mathbf{Z}} |\phi(2^{1/\varepsilon}\, \varepsilon y - k)| \int_{2^{1/\varepsilon}\varepsilon y + c}^{\infty} \frac{dx}{(1+|x-k|)^{2m+2}} \\
&\qquad \leqslant C_1 \frac{1}{(1+|2^{1/\varepsilon}\varepsilon c|)^m} \sum_{k \in \mathbf{Z}} \frac{1}{(1+|2^{1/\varepsilon}\varepsilon y - k|)^m} \frac{1}{(1+|2^{1/\varepsilon}\, \varepsilon(y+c) - k|)^m} \\
&\qquad \leqslant C_2 \frac{1}{(1+|2^{1/\varepsilon}\varepsilon c|)^m}.
\end{aligned}
$$

It follows that $(1+y^2)^{m/2}\, |I_3|$ tends to zero when $\varepsilon \to 0$. In a similar way it can be shown that

$$(1+y^2)^{m/2}\, (|I_1| + |I_4| + |I_5|) \to 0 \qquad \text{when} \quad \varepsilon \to 0.$$

It remains to show that $(1+y^2)^{m/2}\, |I_2| \to 0$ when $\varepsilon \to 0$.

$$(1+y^2)^{m/2}\left|\int_{y-c}^{y+c} E_{\eta(\varepsilon)}(y,x)\Big(\sigma^{(p)}(x)-\sigma^{(p)}(y)\Big)\right| dx$$

$$\leqslant (1+y^2)^{m/2}\,|\sigma^{(p+1)}(\xi)|\,c\int_{y-c}^{y+c}|E_{\eta(\varepsilon)}(y,x)|\,dx$$

$$\leqslant C\frac{(1+y^2)^{m/2}}{(1+|y-c|)^s}\int_{y-c}^{\infty}|E_{\eta(\varepsilon)}(y,x)|\,dx,$$

where $\xi\in(y-c,\,y+c)$. Since the last integral tends to zero when $\varepsilon\to 0$ we conclude that

$$\lim_{\varepsilon\to 0^+}\left\langle\frac{h(\varepsilon y)}{\rho(\varepsilon)},\,\langle E_{\eta(\varepsilon)}(x,y)-\delta(x-y),\,\sigma(x)\rangle\right\rangle=0,$$

and therefore that

$$\lim_{\varepsilon\to 0^+}\left\langle\frac{h_{1/\varepsilon}(\varepsilon x)}{\rho(\varepsilon)},\,\sigma(x)\right\rangle=\lim_{\varepsilon\to 0^+}\left\langle\frac{h(\varepsilon y)}{\rho(\varepsilon)},\,\sigma(y)\right\rangle$$

which completes the proof. ▮

*Remarks.* (1) We say that $h\in\mathscr{S}'$ has *the quasiasymptotics at* $x_0$ (*in* $\mathscr{S}'$) *related to* $c(\varepsilon)$, if there exists $g\in\mathscr{S}'$, $g\neq 0$, such that

$$\lim_{\varepsilon\to 0^+}\left\langle\frac{h(\varepsilon x+x_0)}{c(\varepsilon)},\,\sigma(x)\right\rangle=\langle g(x),\,\sigma(x)\rangle,\qquad \sigma\in\mathscr{S}.$$

Theorems 3 and 4 also hold if *zero* is replaced with $x_0$. In that case the proofs are slightly different as indicated next.

In the same manner as we proved Lemma 1, we can show that the family

$$\left\{\left\langle E_{\varphi(j,\,\varepsilon)}\left(x+\frac{x_0}{\varepsilon},\,y+\frac{x_0}{\varepsilon}\right),\,\sigma(x)\right\rangle,\,\varepsilon\in(0,1);\,\sigma\in\mathscr{S}\right\}$$

is, for every $j\in\mathbf{Z}$, bounded in $\mathscr{S}$, uniformly in $\varepsilon$. Further on, we write

$$h_j(\varepsilon x+x_0)=\left\langle h\left(y+\frac{x_0}{\varepsilon}\right),\,E_j\left(\varepsilon x+x_0,\,y+\frac{x_0}{\varepsilon}\right)\right\rangle,$$

and repeat the proofs of Theorems 3 and 4 step by step, with some obvious modifications.

(2) It is possible, with minor modifications, to obtain analogous statements for the quasiasymptotic behavior in the more dimensional case

for radial distributions, i.e., those depending only on the modulus of the $n$-dimensional variable.

(3) The wavelets have the property of localization, which is somewhat in contradiction with the global character of the quasiasymptotics at infinity. It is an open problem if the distributional behavior at infinity can be characterized by means of wavelet expansions.

(4) It might be of interest to compare other wavelet-type expansions, e.g., wavelet expansion, scaling expansion (see [KKR]) with the quasi-asymptotic behavior.

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